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Extension Research

Designing Control Charts for Minimum Total Quality Costs

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Abstract

The basic control chart consists of sampling from a process over time and charting process measurements. For as long as the measurements fall inside control limits (L), the process is considered in-control. In feed production, it is customary to set the control limits at two standard deviations. This practice does not consider the probability of both Type I and Type II errors in addition to all the costs incurred. The objective of this research was to derive a general methodology to determine optimum sample size (n), sampling period (h), and location of L for an X chart used to monitor production processes. A quality cycle is defined as the time between the start of successive in-control periods. Let A_1 be the average production length while in-control and A_2 when the process has shifted to an out-of-control state. The total cost of quality is given by $C = \{C_0/\lambda + C_1 (-t + nE + h(A2) + \delta_1 T_1 + \delta_2 T_2) + sY/A_1 + W\} \div \{1/\lambda + (1 - \delta_1)sT_0/A_1 - t + nE + h(A_2) + T_1 + T_2\} + \{[a + bn)/h] \times [1/\lambda - t + nE + h(A_2) + \delta_1 T_1 + \delta_2 T_2]\} \div \{1/\lambda + (1 - \delta_1)sT_0/A_1 - t + nE + h(A_2) + T_1 + T_2\}$, with $s = \exp(-\lambda h) / (1 - \exp(-\lambda h))$, $t = [1 - 1 + \lambda h)\exp(-\lambda h)] / [\lambda(1 - \exp(-\lambda h)]$, $A_1 = 1/\alpha$, $A_2 = 1/(1 - \beta)$, $\alpha = \Pr(X - CL > L\sigma/\sqrt{n}) + \Pr(X - CL < -L\sigma/\sqrt{n} < X - CL < L\sigma/\sqrt{n})$, where t = expected time of occurrence of the assignable cause; s = expected number of samples taken while in-control; λ = 1/mean time process is in-control; E = time to sample and chart one item; T_0 = expected search time when false alarm; T_1 = expected time to discover the assignable cause; T_2 = expected time to repair the process; δ_1 = 1 if production continues during searches, 0 otherwise; δ_2 = 1 if production continues during repair, 0 otherwise; C_0 = quality cost/hour while producing in-control; C_1 = quality cost/hour while producing out-of-control; Y = cost per false alarm; W = cost to locate and repair the assignable cause; a = fixed cost per sample; b = cost per unit sampled. The values n*, h* and L* that give the minimum C provides the optimum control design. The procedure is easily implemented on a modern spreadsheet.

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Introduction

In dairy production, as in most farm production systems, a variety of processes must be monitored in order to achieve maximum economic efficiency. Monitoring must be done because although a given process is (hopefully) producing expected outcomes while in-control, it can shift at any point in time to an out-of-control status. Control charts have long been used to monitor processes and to provide signals of out-of-control status (Shewhart, 1931). The basic X control chart consists of sampling from a process over time and charting process measurements. For as long as the measurements fall inside control limits (L), the process is considered in-control. In most farming systems, it is customary to set the control limits at two standard deviations (DeVries *et al.*, 1997). This practice, however, does not consider the respective size of Type I and Type II errors as well as all the costs incurred. The objective of this research was to derive a general methodology to determine optimum sample size (n), sampling period (h), and location of L for an X chart. For this report, the production of pelleted feeds is used as an example, but the method can be applied easily to any renewal reward process.

Methods

To optimize the quality control procedure, the total cost of quality must be derived as a function of n, h, and L. To do so, a quality cycle is defined as the time between the start of successive in-control periods (Figure 1). The costs incurred during the in-control period are: (1) costs from sampling the process, (2) costs of some non-conform products, and (3) costs of false alarms. It is assumed that when the process goes out-of-control, it shifts to a known state and cannot return to an in-control state without intervention. This assumption is not major, considering that Duncan (1971) and Knoppenberger and Grandage (1969) both concluded that a single assignable-cause model with a weighted average shift approximates closely the minimal cost for the multiple cause model. The costs incurred during the out-of-control period are: (1) costs due to sampling, (2) costs due to the increased level of defective products, and (3) costs due to searching for the cause, repairing the system, and downtime.

1. Duration of Cycle.

First, consider the duration of a cycle. It is the sum of the following:

(a) The time until the assignable cause occurs.

Given that this is a memoryless process subject to random shocks, the in-control period is distributed as a negative exponential random variable with mean $1/\lambda$. If production continues during searches, the average time for occurrence

Figure 1. Graphical representation of in-control and out-of-control states of a process.

of the assignable cause is simply $1/\lambda$. If production ceases during the search, then it gets a little messier. Let T_0 be the expected search time for a false alarm. Then, the expected time spent searching during the false alarm is T_0 times the expected number of false alarms = $T_0 (s/A_1)$, where A_1 is the average run length while in control and s is the expected number of samples taken while in control. A few steps of algebraic derivations show that $s = e^{-\lambda h}/(1 - e^{-\lambda h})$, and $A_1 = 1/\alpha$ where α = Probability (out-of-control signal | process is in control). Note that A_1 depends only on the assumed underlying distribution and the control limit L. There is one elegant way of combining both conditions. Let $\delta_1 = 1$ if production continues during searches and $\delta_1 = 0$ if production ceases during searches. Then, the expected time until the assignable cause occurs is:

(1) $1/\lambda + (1 - \delta_1)\, s\, T_0/A_1$

(b) The time until the next sample is taken.

Now, let T be the expected time of occurrence of the assignable cause, given that it occurs between the i^{th} and (i + 1)st samples. By derivation, we get

(2) $T = [1 - (1 - \lambda h)e^{-\lambda h}] / [\lambda (1 - e^{-\lambda h})]$

and the expected time between the occurrence of the assignable cause and the next sample equals

$$h - T$$

(c) The time to analyze the sample and chart the result.

Let E be the expected time to sample and chart one item. Note that E is often taken as 0, but it can be quite large if samples have to be sent to a laboratory. For a sample of n items, the time to analyze the sample and chart the result is given by:

(3) nE

(d) The time until the chart gives an out-of-control signal.

The expected time until an out-of-control signal occurs is given by $h\,(A_2 - 1)$, where A_2 is the average run length when the process has shifted to an out-of-control state. If the sample statistics are independent, then $A_2 = 1/(1 - \beta)$, where β = Probability (in-control signal | process is out-of-control). Note that A_2 depends on the underlying distribution, the control limit L, the sample size n, and the extent of the shift Δ when the assignable cause occurs.

(e) The time to discover the assignable cause and repair the process.

Let T_1 be the expected time to discover the assignable cause and T_2 be the expected time to repair the process. Then the expected time to detect a shift, discover the assignable cause, and repair the process equals:

(4) $h\,(A_2 - 1) + T_1 + T_2$

Combining equations (1) through (4), we now know that:

(5) Expected cycle time =
$1/\lambda + (1 - \delta_1)\, s\, T_0/$
$A_1 - T + nE + h\,(A_2) + T_1 = T_2$

Cost Function

The costs per cycle are incurred for defective production while in-control as well as out-of-control, for false alarms, for location and repair of the assignable cause, and for sampling and inspection.

(a) Cost per cycle due to defective products.

Let C_0 and C_1 ($> C_0$) be the costs per unit of time (hours or days) due to nonconformities produced while the process is in-control and out-of-control, respectively. Then the expected cost per cycle due to defective products equals:

(6) $C_0 / \lambda + C_1 (- T + nE + h(A_2) + \delta_1 T_1 + \delta_2 T_2.$

(b) Cost per cycle due to false alarms, locating, and repairing.

Let Y be the cost per false alarm. This includes the cost of searching and testing for the cause plus the cost of downtime if production ceases during the search. Also, let W be the cost for locating and repairing the assignable cause when one exists. Then, the expected cost for false alarms and locating and repairing the true assignable cause is:

(7) $sY / A_1 + W$

(c) Cost per cycle for sampling and inspection.

Let a be the fixed cost per sample and b the cost per unit sampled. Then the expected cost for sampling and inspection is given by (a + bn) (time producing) / h. The expected cost per cycle for sampling equals:

(8) $(a + bn) (1/\lambda - T + nE + h (A_2) + \delta_1 T_1 + \delta_2 T_2) / h$

Combining (6), (7), and (8), we get the total quality cost per cycle as:

(9) $C_0 / \lambda + C_1 (- T + nE + h(A_2) + \delta_1 T_1 + \delta_2 T_2) + sY / A_1 + W + (a + bn) \times (1/\lambda - T + nE + h (A_2) + \delta_1 T_1 + \delta_2 T_2) / h$

Because cycle lengths are variable and are functions of h, n, and L, we must express the cost function per unit of time (hours, days), not per cycle. This is easily done by dividing equation (9) by equation (5) to yield the cost per hour, C.

(10) $C = \{C_0 / \lambda + C_1 (- Y + nR + h(A_2) + \delta_1 T_1 + \delta_2 T_2) + sY / A_1 + W\} + \{1/\lambda + (1 - \delta_1) s T_0 / A_1 - T + nE + h(A_2) + T_1 + T_2\} + \{[(a + bn)/h] \times [1/\lambda - T + nE + h(A_2) + \delta_1 T_1 + \delta_2 T_2]\}$

All that remains is the derivation of A_1 and A_2 in equation (10). For an X chart, it is assumed that the observations are iid normal with mean equal to the centre line (CL) and standard deviation σ. Thus:

$\alpha = Pr (X - CL > L\sigma/\sqrt{n}) + Pr (X - CL < -L\sigma/\sqrt{n}) = 2\, \phi(-L)$

where ϕ is the standard normal distribution.

When the process is out-of-control, the observations are iid normal with mean CL + Δσ, so:

$\beta = Pr (-L\sigma/\sqrt{n} < X - CL < L\sigma/\sqrt{n}) = \phi(L - \Delta\sqrt{n}) - \phi (-L - \Delta\sqrt{n})$

Substitution of 1/α for A_1 and 1/(1-β) for A_2 in (10) gives the quality cost per hour for any process to be monitored by an X chart.

Minimizing the Cost Function

In a given process, the cost per hour is given by (10) and is a function of three quality control variables — the sample size (n), the sampling period (h), and the control limit (L). Function (10) is highly nonlinear in each of the three parameters. Therefore, nonlinear optimization techniques must be used to identify the optimum n, h, and L leading to the minimum cost. Thus, a combination of Newton's gradient method and a golden section search (Himmelblau, 1972) is used successfully to solve the problem of optimum quality control in feed-production processes.

An Example

A feed mill produces 8.5 tons per hour of pelleted feeds. Periodic samples are taken and the pellet durability index (PDI) is measured (an indicator of pellet quality). A PDI of 96% is desired to prevent shipments of products susceptible to damage during transport and handling. The sampling process costs $4.25 per sample (labor and equipment). The cost of a defective product averages $100 (increased frequency of complaints, cost to replace defective product, etc.).

Historical data indicate that the process produces 1.36% defective loads when in-control and 11.3% defective loads when out-of-control. The process stays in-control an average of 50 hours. When the process is out-of-control, the pellet mill must be stopped, the die pulled out, the steam system and the pellet mill checked. This takes approximately 45 minutes with a repair crew cost of $23 per hour and a downtime cost of $22 per minute.

In this example, $\lambda = 1/50$, $\Delta = (0.113 - 0.0136) / [(0.0136)(0.9864)]^{1/2} = 0.86$, $E = T_0 = T_1 = 5/60$, $T_2 = 45/60$, $\delta_2 = 0$. The cost per hour while in control, C_0, equals $100/defective loads x 8.5 tons/hour x a defective rate of 0.0136 which equals $11.56/hour. Likewise, C_1 = $100 x 8.5 x 0.113 = $96.05/hour. The cost per false alarm, Y, equals the cost for the assignable cause, equals downtime plus repair cost: 45 x $22 + (45/60) x $23 = $1,007.25. Finally, the fixed cost per sample, a, equals 0, and the cost per unit sampled, b, equals $4.25.

Using this optimization procedure, the best sampling plan is n = 20, h = 2.88, and L = 3.336. That is, take a sample of size 20 every 2.88 hours and search for the cause of the quality slip if the mean of the 20 samples is 3.336 units under the process average. The expected cost per hour is $23.72 with $A_1 = 33.5$ and $A_2 = 1.5$. This corresponds to an α error of 0.03 and a β error of 0.32.

Ishikawa (1976) provided guidelines that correspond to a sampling plan of n = 250, h = 8, and L = 3, resulting in an hourly quality cost of $53.29. The difference between this plan and the optimal plan is $29.57 per hour or approximately $220,000 annually for a pellet mill operating six days per week on a three-shifts per day schedule.

Conclusions

In this research, a quality cost function applicable to any control chart of the form developed by Shewhart (1931) was derived. This function depends on 12 cost and time parameters that describe the process, two indicator variables that determine if production continues during search or repair, and three design parameters that describe the charting procedure. The minimization of this function over the choice of design parameters leads to the most economical control charts. Considerable cost savings can be achieved without changing the fundamental control chart format.

Results will be used to derive optimal quality-control procedures to monitor somatic cell counts and milk urea nitrogen in commercial dairy herds.

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